Contributions to credit risk

Optimisation of credit portfolios requires that risk contributions be quantified. However, there has been disagreement over which of three popular tail risk measures should be used. Here, Alexandre Kurth and Dirk Tasche offer a way forward, showing how to calculate all three measures in the context of CreditRisk+, and then applying the calculation to a set of sample portfolios, with interesting results.

From a theoretical point of view, there is no need to assign risk contributions to the parts of a portfolio. This is because no rational investor would be happy with a sub-optimal portfolio in the sense of best possible return-risk ratios. Therefore, the assumption of already-optimised portfolios predominates in the scientific literature. Of course, in the case of an optimal portfolio, surveillance of sub-portfolios is dispensable since, by definition, such a portfolio cannot be improved.

Nevertheless, in practice, particularly in credit practice, optimal portfolios are rare. Even if the causes of bad performance in a credit portfolio have been located, because of limited liquidity it will in general be impossible to optimise the portfolio in one step. Indeed, credit portfolio management is a sequence of small steps rather than a one-strike business. However, steering a portfolio step by step requires detailed risk diagnoses. The key to such diagnoses is assigning appropriate risk contributions to the sub-portfolios.

In a recent article, Koyluoglu & Stoker (2002) reviewed several approaches to the problem of how to define sensible risk contributions. Here, we investigate the ‘continuous marginal contribution’ approach for the CreditRisk+ (CSFP, 1997) credit portfolio model. More exactly, we determine analytically the contributions of each obligor to value-at-risk and expected shortfall (CSFP, 1997) credit portfolio model.

The analytical results presented in this article show that CreditRisk+ offers only one extra run for each segment in the CreditRisk+ model. As a byproduct, a decreasing list of ES contributors, even for large portfolios, may be found. Apart from this, the contribution to ES is a sequence of small steps rather than a one-strike business. However, steering a portfolio step by step requires detailed risk diagnoses. The key to such diagnoses is assigning appropriate risk contributions to the sub-portfolios.

This article first outlines the main features of the CreditRisk+ model. Then we introduce some aspects of risk measures. The formulas for the VAR and ES contributions are then derived for the one-segment case, and are extended to several segments. The article ends with an example where the different risk measures are compared based on independent, uncorrelated segments, as well as correlated segments. The appendix sketches the proof of the main statement given by equation (13).

Short overview of CreditRisk+

We will briefly outline the main aspects and relations of the CreditRisk+ model in the one-segment case. The starting point of this credit portfolio model is the following equation for the random portfolio loss $L$ over all obligors $A$:

$$L = \sum_A I_A N_A$$

where $I_A$ is interpreted as an indicator variable describing the default event of $A$, that is, $I_A = 1$ if $A$ defaults and $I_A = 0$ otherwise, and $N_A$ is the exposure net of recovery.¹ Let us denote the (unconditional) default probability of obligor $A$ by $p_A$, that is, $p_A$ is the mean of $I_A$. The dependence between obligors is incorporated by a common risk factor $S$, which will later be assumed to be gamma-distributed. The mean of the variable $S$ is equal to $\mu = \sum A p_A$, and its volatility is denoted by $\sigma$. Then, conditional on $S$, the expectation $^*$ of $I_A$ is $Sp_A/\mu$. This default scaling factor $S$ reflects the intensity of the number of default events in the economy (Bürgisser et al, 1999).

The expected loss $EL$ and the standard deviation $UL$ of the loss variable $L$ can now be derived (Bürgisser, Kurth & Wagner, 2001):

$$EL = \sum_A p_A N_A, \quad UL^2 = \frac{\sigma^2}{\mu} EL^2 + \sum_A \left[p_A \left(1 + \frac{\sigma^2}{\mu}\right)\right] \left[p_A\right]^2$$

There is a slight difference to the formula in CSFP (1997, equation 118). This is caused by the assumption made there that the default is modelled by a conditionally Poisson distributed random variable while here defaults are Bernoulli events.² However, to understand the loss distribution, and not just the first and second moments, it is convenient to adopt the Poisson approximation and make use of the probability generating function³ $G(z)$, which is defined as a power series of the form:

$$G(z) = \sum_{n=0}^{\infty} p(n) z^n$$

Here $p(n)$ is the probability of losing the amount $n$ and $z$ is a formal variable. To calculate the coefficients $p(n)$ we make the following assumptions. First, the exposures (net of recovery) are banded.¹ Second, the default events $I_A$, conditional on $S$, are approximated by independent Poisson variables. Third, the common risk factor $S$ is gamma-distributed with mean $\mu = \sum_A p_A$ and volatility $\sigma$.

With these assumptions, it turns out that the probability-generating function of the portfolio loss satisfies the following relation (see CSFP, 1997, equation 68, and Bürgisser, Kurth & Wagner, 2001, equation 13):

$$G(z) = \left(1 - \frac{\sigma^2}{\mu} \left(Q(z) - 1\right)\right)^\mu$$

where $\alpha = \mu^2/\sigma^2$, and $Q(z) = \mu^{-1} \sum A p_A z^A$ denotes the so-called portfolio polynomial. Equation (4) can be used to derive a recursive condition on the coefficients $p(n)$, which is due to Panjer (1980, 1981) (see also CSFP, 1997).

In this article, the loss severities are assumed to be constant for each obligor. For an extension to variable severities, see Bürgisser, Kurth & Wagner (2001).

¹ Sp/A may also be regarded as the conditional default probability of obligor $A$ given state $S$ of the economy. However, in the model under consideration, Sp/A may take values greater than one. Thus, the interpretation as probability is to an approximative sense.

² The derivation of the UL-formula in Credit Suisse Financial Products (1997) makes use of probability generating functions as the exposures are banded. However, this is not necessary as shown in Bürgisser, Kurth & Wagner (2001, theorem 1).

³ For an introduction of probability generating functions and their properties, see Fisz (1960).

⁴ That is, the exposures (net of recovery) $v_A$ can be assumed to be integer values.

⁵ In this article, the loss severities are assumed to be constant for each obligor. For an extension to variable severities, see Bürgisser, Kurth & Wagner (2001).
factors is, to the case where the economy is described by a set of systematic risk factors $S_1, \ldots, S_N$. Moreover, under the assumption of independent segments with gamma-distributed factors, the extension for (5) is given in CFSF (1997, equation 77).

How to measure risk

A risk measure is a metric measuring the uncertainty of the portfolio loss. If a portfolio is given by an element of the set $P = \{ (v, A\text{ obligor}) \}$ where $v_A$ represents the exposure net of recovery of $A$, then the portfolio loss is given by (1), that is, $L = L(v) = \sum_i I_{v_A, i}$. Formally, a risk measure is described by a function $p : P \rightarrow \mathbb{R}$, and for every portfolio $v \in P$, the number $p(v)$ is the risk of $v$.

There are two particularly popular examples of risk measures. One is the standard deviation (or unexpected loss, UL) which is popular mainly for its computational simplicity. The level $UL$ is defined by (1). The other is the VAR at level $\alpha$ and the conditional expectation of $\frac{\delta}{\alpha}$:

$$\frac{\partial \text{VAR}_\alpha(v)}{\partial \alpha} = E\left[I_A | L(v) = q_\alpha(L(v))\right]$$

(11)

as the contribution of obligor $A$ to portfolio VAR in the general case. With $UL$, the additivity property holds with definition (12) of the VAR contributions, that is, $\sum C_A^{UL} = VAR_R$.

Under the assumptions above ('Short overview of CreditRisk+') that have led to (4), one can show (see the appendix for a proof in the one-segment case and Haaf & Tasche, 2002, for a different, general proof) that the expectation on the right-hand side of (12) is given by:

$$E_A\left[I_A | L = q_\alpha(L)\right] = P_A \frac{Pr_{q_\alpha}(L = q_\alpha(L)) - v_A}{Pr_{q_\alpha}(L = q_\alpha(L))}$$

(13)

where subscript $\alpha$ means that expectation and probability stem from the original generating function (4) whereas subscript $\alpha + 1$ means that the probability has to be derived from (4) with $\alpha = \mu^2/\sigma^2$ replaced by $\alpha + 1 = \mu^2/\sigma^2 + 1$.

Recall that $E_A[I_A | L = q_\alpha(L)]$ in (13) is an approximation for the probability that obligor $A$ has defaulted conditional on the event that the sum of losses equals $q_\alpha(L)$. By the definition of conditional probability, $Pr_{q_\alpha}(L = q_\alpha(L) - v_A)$ can be interpreted as an approximation of the probability of the sum of losses equaling $q_\alpha(L)$ conditional on obligor $A$ having defaulted. Note that $E_A[I_A | L = q_\alpha(L)] = 0$ can happen (see Haaf & Tasche, 2002, for a detailed discussion of this case). Below, the formula for the multi-segment case is provided.

Equations (12) and (13) show that in CreditRisk+ the VAR contributions can be calculated by running the Panjer algorithm (see equation (5)) twice: once with $\alpha = \mu^2/\sigma^2$ and again with $\alpha + 1 = \mu^2/\sigma^2 + 1$. The setup of this second run is given by scaling all $P_A$ with the factor $1 + \alpha^{-1}$, and the default volatility $\delta$ with $\sqrt{1 + \alpha^{-1}}$.

Martin, Thompson & Browne (2001) took (11) as the point of departure for approaching VAR contributions by saddlepoint approximation. In the CreditRisk+ setting, the technique used to arrive at (11) also yields simple formulas for this approach (Haaf & Tasche, 2002).

Contributions to expected shortfall

Via the decomposition:

$$E[L(v) | L > q_\alpha(L(v))] = \sum_i v_i E[I_i | L > q_\alpha(L(v))]$$

(14)

equation (9) suggests the definition of ES contributions as:

$$C_A^{ES}(v) = v_A E[I_A | L > q_\alpha(L(v))]$$

(15)

This definition can be justified in a similar way as the definition of $C_A^{UL}$ in (12).

From (15), we know how to calculate $E[I_i | L > q_\alpha(L(v))]$ in the CreditRisk+ model. It is not hard to see that a similar formula obtains for $E[I_i | L > q_\alpha(L(v))]$.
A. Specification of the sample portfolio

<table>
<thead>
<tr>
<th>Exposure per obligor (in Sfr million)</th>
<th>Segment 1</th>
<th>Segment 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Number of obligors</td>
<td>10,000</td>
<td>10,000</td>
</tr>
<tr>
<td>Probability of default for each obligor</td>
<td>0.5%</td>
<td>1%</td>
</tr>
<tr>
<td>Exposure in class (in Sfr million)</td>
<td>10,000</td>
<td>10,000</td>
</tr>
<tr>
<td>Exposure (% of total)</td>
<td>16.9%</td>
<td>16.9%</td>
</tr>
</tbody>
</table>

\[
E_a \left[ I_A | L > q_A(L) \right] = p_A \frac{Pr_{q_A+1}[L > q_A(L) - q_A]}{Pr_{q_A}[L > q_A(L)]}
\]

where the subscripts \( \alpha \) and \( \alpha + 1 \) have the same meanings as in (13). Note that (16) can be calculated in finitely many steps since \( Pr_{q_A+1}[L > q_A(L) - q_A] = 1 - Pr_{q_A}[L > q_A(L)] \). This observation can be transferred to ES itself. The determination of the actual expected shortfall by the recursion equation (5) requires that infinitely many elements are calculated. But applying the sum of all contributions \( S \sum C_A \delta \) using equation (16) yields the expected shortfall precisely without knowing the tail beyond the corresponding 6-quantile loss.

Extension to several segments

There are two ways of extending the concept of the contributions to expected shortfall (and VAR) to several segments. First, one can make use of \( N \) independent segments described by an \( N \)-tuple of independent systematic risk factors \( S_1, \ldots, S_N \) which are gamma-distributed. In this case, one has to introduce factor loadings \( \omega_j \) to be the portion of the default probability \( p_A \) allocated to segment \( j \). This means that \( \omega_{jA} = 1 - \sum_{j=1}^{\alpha-1} \omega_j \) is the specific weight of \( A \) accounting for the idiosyncratic default risk.

And, conditional on \( S_1, \ldots, S_N \), the expectation of \( I_s \) is given by:

\[
E[I_s | S_1, \ldots, S_N] = p_A \frac{\omega_{jA} \mu_0 + \sum_{j=1}^{\alpha} \omega_j \mu_j}{\sum_{j=1}^{\alpha} \omega_j} \tag{17}
\]

with \( \mu_i > 0, 1 \leq j \leq N, \) being constant.

The extension of the VAR contribution, as in equation (13), to \( N \) independent segments has the following form:

\[
E_a \left[ I_A | S_1, \ldots, S_N \right] = \frac{\sum_{j=1}^{\alpha} \omega_j Pr_{\alpha+1} [L = q_A(L)]}{\sum_{j=1}^{\alpha} \omega_j} \tag{18}
\]

where \( \alpha(j) = (\alpha_1, \ldots, \alpha + 1, \ldots, \alpha_N) \). The expression \( Pr_{\alpha+1}[L = q_A(L) - q_A] \) is obtained by evaluating the corresponding element of the multivariate version of equation (5) (see CSFP, 1997, equation 77 for \( N \) independent segments) where the jth gamma distribution is specified by the pair \( (\alpha_j + 1, \beta_j) \). Similarly, by replacing the equality signs in \( Pr_{\alpha+1} \) with inequalities \( > 0 \) one obtains the corresponding formula for the contribution to ES for \( N \) independent segments.

However, a segmentation with independent segments is rather restricting. A dependence structure between the obligors is achieved by apportioning the default probabilities to the orthogonal factors as done with \( \omega_{jA} \). This is rather arbitrary, and therefore creates an additional source of uncertainty. These factors typically represent the default behaviour of industrial sectors or geographical areas, and, as such mostly incorporate significant correlations between them.

To get around the restriction of independent segments, one can introduce the correlation matrix of the \( N \) gamma variables \( S_1, \ldots, S_N \) and can match the UL formula of the loss distribution for these \( N \) segments with the corresponding one-segment UL formula (see equation (2)). This procedure is introduced in Burgisser et al (1999, equation (13)). Now, the portfolio is unified in one segment, and the contributions to VAR and ES are determined as in the previous sections following equations (13) and (16). This method is simple and fast. However, it destroys information about dependence and the contributions tend to the middle since the matched dependence structure is given by one volatility, which turns out to be some average of the segment volatilities.

Orthogonalisation of segment correlations does not represent an alternative because the process of orthogonalisation is not unique. It can be shown by examples that significant differences in VAR, ES and their contributions show up.

Numerical example

We want to show that the risk contributions according to the expected shortfall ES and VAR may significantly differ from the risk contributions according to the standard deviation. We will do so by using a wholesale bank portfolio that includes a retail sub-portfolio as well as commercial loans of various sizes. Table A describes the portfolio, which is characterised by the loan sizes given in the first line.

We assume that all loan exposures are given net of recovery. As indicated in table A, we further assume a segmentation: the loans of exposure Sfr1 million form segment one (the retail segment) and all others build the second segment (commercial segment).

Using the portfolio in table A, we present three different cases for the computation of ES, VAR, UL and the corresponding contributions. The first case is characterised by the segment volatilities \( \sigma_{s_1}, \sigma_{s_2} \) of default rates and independence of segments one and two, where we use equation (18) for determining the VAR and ES contributions. In the second run, we take the same segment volatilities as in the first case, but now we use the method of moment matching' described at the end of the previous section. In the last case, we introduce correlations between both segments and again use the moment-matching method to calculate the contributions. Note that the covariance of 0.21 in the last case corresponds to a 70% correlation between the segments. The segment volatilities \( \sigma_j \) are given in terms of their normalised means, for example, the volatility \( \sigma_1 \) of defaults in segment one for all three cases is \( \sigma_1 = \sqrt{0.16} \) the expected number of defaults in segment one, hence \( \sigma_1 = \sqrt{0.16} \times 0.75 \times 10,000 \). Table B gives the exact description for these three cases.

Observe that independence does not necessarily lead to the lowest possible risk measure. Tables C to E list the segment contributions to the measures of risk for the three cases described above: standard deviation UL, 99-percentile VAR and ES at 99%. To compare these, we give the ratios of the relative contributions per exposure class for the three risk measures.

Note that in all cases the retail loans (Sfr1 million), and particularly the large lumpy loans, contribute more to ES (and to VAR) than they do to UL, which shows that ES measures concentration more sensitively. For the large loans this is quite intuitive, since once a tail loss above ES is reached, the large (lumpy) loans more often contribute by their defaults. But apparently, a significantly large amount of small loans also suffices to produce

\[\text{Here we assume zero correlation, whereas in case 1 we assume independence. Note that uncorrelatedness of two gamma distributions does not imply their stochastic independence.}\]
a large loss. In case one, the contributions for VAR and ES almost coincide for loans in segment one, whereas they increasingly diverge for increasing loan amounts.

Note that in the ‘zero-correlation’ case (table D) the UL-contributions are calculated based on the corresponding covariance matrix (see Bürgisser et al, 1999, equation (15)) whereas the ES and VAR contributions are calculated based on the matched one-segment approach. This causes the large differences in the relative contributions of segment one. Calculating the UL-contributions based on the matched one-segment approach would lead to a ratio in the range of one for $C(ES)/C(UL)$ in segment one. Applying one volatility for the entire portfolio causes a shift of the contributions to the ‘average’. The exposure classes one and 10, both with $p_A = 1\%$, have the same total exposure and very similar contributions for the ‘zero-correlation’ and ‘correlation’ cases. But in the ‘independence’ case their contributions differ greatly (for example, 105 versus 312 for ES).

Introducing correlations, we see a large increase in the UL-contributions in segment one whereas they remain quite stable in the other segment. The ES contributions increase by about 10\%, except for both of the largest exposure classes, where a decrease of approximately 10\% is realised.
changes of VAR contributions are similar but smaller, with the exception of the VAR contribution reduction in exposure class 50.

Conclusion
We have introduced an analytical approach to calculate contributions to VAR and ES of a credit loss distribution in the CreditRisk+ framework. The formulas we have derived are easy to implement. The time taken to calculate these contributions for all obligors is the same as for calculating the loss distribution in the moment-matching case of CreditRisk+.

The results show that VAR and ES contributions may differ significantly from UL contributions, in particular for large exposures. Moreover, ES is especially sensitive to concentrations coming from large loans, which represent a source for stress losses in a loan portfolio. So if a bank wants to actively manage its loan portfolio, whose performance is measured (among others) by tail losses, it should take into account that such tail events are composed differently when measured by ES instead of by standard deviation (or a multiple thereof).

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Appendix

Here we sketch the proof of equation (13). Denote as usual by \( I_t \) the indicator function of the event \( L = t \), that is, \( I_t = 1 \) if \( L = t \) and \( I_t = 0 \) otherwise. By definition, we have:

\[
E_\alpha[I_t(L)] = \frac{E_\alpha[I_t(L)I_{(t+1)}(L)]}{Pr_\alpha(L = q_\alpha(L))} \quad (19)
\]

We will now calculate the generating function of the sequence \( t \mapsto E_\alpha[I_t I_{(t+1)}] \), which is defined as the function:

\[
z \mapsto E_\alpha[I_z I_{z+\mu}] = \sum_{t=0}^{\infty} E_\alpha[I_t I_{t+\mu}] \quad (20)
\]

In the CreditRisk+ model, conditional on the intensity \( S, I_t \) is approximated by a Poisson variable with intensity \( p_\alpha q_\alpha S \) and the default events are assumed to be conditionally independent, so we obtain:

\[
E_\alpha[I_t I_{t+\mu}] = \sum_{t=0}^{\infty} k E_\alpha[I_t = k \cdot \sum_{v \leq k} v I_{v \mu} = t - \nu_k] = \sum_{t=0}^{\infty} k E_\alpha[I_t = k \cdot \sum_{v \leq k} v I_{v \mu} = t - \nu_k[S]]
\]

\[
= \frac{p_\alpha}{\mu} \sum_{k=0}^{\infty} E_\alpha[SP_{\alpha}(I_t = k | S)Pr_\alpha(\sum_{v \leq k} v I_{v \mu} = t - \nu_k[k + 1]|S)]
\]

Equation (20) implies:

\[
E_\alpha[I_z I_{z+\mu}] = \frac{p_\alpha z^\nu_\alpha}{\mu} E_\alpha[\frac{\alpha}{\mu} S(z)]
\]

From (22) and (21), we conclude:

\[
E_\alpha[I_z I_{z+\mu}] = \frac{p_\alpha z^\nu_\alpha}{\mu} E_\alpha[S(z)]
\]

Recall that both sides of (23) can be written as power series. Hence, we can conclude that:

\[
E_\alpha[I_z I_{z+\mu}] = \frac{p_\alpha z^\nu_\alpha}{\mu} E_\alpha[\frac{\alpha}{\mu} S(z)]
\]

for \( t = 0, 1, 2, \ldots \). Choosing \( t = q_\alpha(L) \) now proves (13). ■

REFERENCES

Acerbi C and D Tasche, 2002
On the coherence of expected shortfall
Journal of Banking & Finance 26(7), pages 1,487–1,503

Artzner P, F Delbaen, J-M Eber and D Heath, 1999
Coherent measures of risk
Mathematical Finance 9(3), pages 203–228

Bürgisser P, A Kurth and A Wagner, 2001
Incorporating severity variations into credit risk
Journal of Risk 3(4), pages 5–31

Bürgisser P, A Kurth, A Wagner and M Wolf, 1999
Integrating correlations
Risk July, pages 57–60

CSFP, 1997
CreditRisk+: a credit risk management framework

Fisz M, 1980
Probability theory and mathematical statistics
John Wiley & Sons, third edition

Gourieroux C, J-P Laurent and O Scaillet, 2000
Sensitivity analysis of values at risk
Journal of Empirical Finance 7, pages 225–245

Haaf H and D Tasche, 2002
Credit portfolio measurements
GARP Risk Review 7, July/August, pages 43–47

Hallerbach W, 2003
Decomposing portfolio value-at-risk: a general analysis
Journal of Risk 5(2)

Koyluoglu H and J Stoker, 2002
Honour your contributions
Risk April, pages 90–94

Lemus G, 1999
Portfolio optimization with quantile-based risk measures

Martin R, K Thompson and C Browne, 2001
VAR: who controls and how much?
Risk August, pages 99–102

Panjer H, 1980
The aggregate claims distribution and stop-loss reinsurance
Transactions of the Society of Actuaries 32, pages 523–545

Panjer H, 1981
Recursion evaluation of a family of compound distributions
ASTIN Bulletin 12, pages 22–26

Reckzell R and S Uryasev, 2002
Conditional value-at-risk for general loss distributions
Journal of Banking & Finance 26(7), pages 1,443–1,471

Tasche D, 1999
Risk contributions and performance measurement

Yamal Y and T Yoshihara, 2001a
On the validity of value-at-risk: comparative analyses with expected shortfall
IMES discussion paper 2001-E-4, Bank of Japan

Yamal Y and T Yoshihara, 2001b
Comparative analyses of expected shortfall and VaR: their estimation error, decomposition and optimization
IMES discussion paper 2001-E-12, Bank of Japan